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Albert-Ludwigs-Universität Freiburg
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Topology

Problem Sheet 9

Deadline: 25 June 2024, 15h

Exercise 1 (9 Points)**.**

Consider a set X equipped with the discrete topology.

a) Given x_0 in X, compute the fundamental group $\pi_1(X, x_0)$.

A topological space Y is *contractible* if the identity map Id_Y is homotopic (see Sheet 8, Exercise 4) to some constant map $y \mapsto y_0$, that is, there is a homotopy $H : Y \times [0,1]$ such that $H(y,0) = y$ and $H(y,1) = y_0$.

b) Show that a contractible topological space must be path connected.

c) Show that a contractible topological space is simply connected.

Hint: Given a loop γ at y_0, show that $H(\gamma(s), t)$ can be *conjugated* to obtain a obtain a path homotopy from γ to the constant path C_{y_0}.

d) Is a convex subset of $\mathbb{R}^n$ with the subspace topology relative to the euclidean topology contractible?

e) If the discrete space X has at least two points, is X contractible?

Hint: Which topological properties does the interval $[0,1]$ fulfill?

Exercise 2 (5 Points)**.**

Consider $\mathbb{R}^n$ with the euclidean topology.

a) Show that the linear maps induced by any two $n \times n$ matrices A and B are homotopic by a homotopy H such that the function $H(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ is given by an $n \times n$ matrix for every t in $[0,1]$.

b) If A and B are both invertible, is $H(\cdot, t)$ always invertible? (for any possible choice of such a homotopy H?)

Exercise 3 (6 Points)**.**

Consider a topological group $(G, \cdot)$, as in Exercise 3 of Sheet 3, as well as two loops α and β based on the neutral element 1_G.

a) Produce an explicit homotopy H_1 between α and $\alpha \star \mathrm{C}_{1_G}$, where C_{1_G} is the constant loop.

b) Produce an explicit homotopy H_2 between β and $\mathrm{C}_{1_G} \star \beta$.

c) Show that $H(s,t) = H_1(s,t) \cdot H_2(s,t)$ is a homotopy between the loop $t \mapsto \alpha(t) \cdot \beta(t)$ and $\alpha \star \beta$, where $\cdot$ denotes the group law on G.

d) Conclude that the fundamental group $\pi_1(G, 1_G)$ is abelian, even if G is not commutative.

Hint: Notice that the constant loop at 1_G is the neutral element for both multiplications.

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im entsprechenden Fach im Keller des mathematischen Instituts.